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A NOTE ON DIELECTRIC LENSES

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In the following we consider dielectric lenses with spherical or cylindrical symmetry, in which the index of refraction N is a function of the distance r from the center of the sphere or from the axis of the cylinder. Only the geometrical properties of optical ray paths are investigated; amplitude, phase and polarization of the electromagnetic wave are neglected. In the spherical case, the problem always reduces to the consideration of ray paths in a plane passing through the center of the sphere; in the cylindrical case, we shall limit ourselves to the case of rays lying in a plane perpendicular to the cylinder axis.

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The following assumptions are made:

1) the radius of the sphere, or of the cylinder, is equal to unity;
2) the index of refraction N(r) of the dielectric material relative to the surrounding medium is finite, except possibly at $r=0$, and $N(1) = 1$;
3) N(r) is continuous with its first derivative in $0 < r < 1$.

The geometry of the problem is the same for both spherical and cylindrical lenses, and is illustrated in Fig. 1. An optical ray enters the lens at P_1 with an angle of incidence α, it is smoothly deviated along the path $P_1 P_2$, and leaves the lens at P_2. Let us indicate by R the optical length of the path $P_1 P_2$, by θ its angular extension as seen from the center $r = 0$, and by δ the deviation of the ray at P_2 from the direction of incidence; it is:

$$\delta = \theta + 2\alpha - \pi \quad . \tag{1}$$

Following the usual procedure (see e.g. Toraldo di Francia, 1957 and Huynen, 1958), we set

$$r^2 N^2 = 1-\xi^2 \quad , \tag{2}$$

and introduce a new function $f(\xi)$ satisfying the relation:

$$2N^2 r\, dr = -f(\xi) d\xi \; . \tag{3}$$

Then it is:

$$R = \int_0^{\cos\alpha} \frac{f(\xi)d\xi}{\sqrt{\cos^2\alpha - \xi^2}} \quad , \tag{4}$$

$$\theta = \sin\alpha \int_0^{\cos\alpha} \frac{f(\xi)d\xi}{(1-\xi^2)\sqrt{\cos^2\alpha - \xi^2}} \quad , \tag{5}$$

$$\ln r = -\frac{1}{2}\int_0^{\xi} \frac{f(\eta)d\eta}{1-\eta^2} \quad . \tag{6}$$

Relations (2) and (6) give N as a function of r, in parametric form.

Formulas (4), (5) and (6) have been employed by various authors. Huynen (1958) has discussed the case in which

$$f(\xi) = a_0 + a_1 \xi \quad , \tag{7}$$

where a_0 and a_1 are constants.

In this note, the expressions for R, θ and $\ln r$ are derived for the case in which $f(\xi)$ is an analytic function of ξ in a neighborhood of $\xi = 0$. It is then pointed out that restrictions have to be placed on the coefficients of the power series representing $f(\xi)$, if one wants to have solutions with physical significance. In particular, the limitations on a_0 and a_1 for the simple case (7) are investigated in detail.

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Let us split $f(\xi)$ into its even and odd components:

$$f(\xi) = f_e(\xi) + f_o(\xi) \quad , \tag{8}$$

where:

$$f_e(\xi) = \frac{f(\xi)+f(-\xi)}{2} \quad , \qquad f_o(\xi) = \frac{f(\xi)-f(-\xi)}{2} \quad . \tag{9}$$

Assume that for the range of values of ξ of interest to us, f_e and f_o may be represented by the uniformly convergent power series:

$$f_e(\xi) = \sum_{m=o}^{\infty} a_{2m}\xi^{2m} \quad , \qquad f_o(\xi) = \sum_{m=o}^{\infty} a_{2m+1}\xi^{2m+1} \quad , \tag{10}$$

where a_{2m} and a_{2m+1} are constants.

Substituting expansions (10) into relations (4), (5) and (6) and integrating term by term, it is found that

$$R = \frac{\pi}{2}\left\{a_o + \sum_{m=1}^{\infty} \frac{(2m-1)!!}{m!\,2^m} a_{2m}(\cos\alpha)^{2m}\right\} +$$

$$+ \sum_{m=o}^{\infty} a_{2m+1}(\cos\alpha)^{2m+1} \sum_{h=o}^{m} \frac{(-1)^h}{2h+1}\binom{m}{h} \quad , \tag{11}$$

$$\theta = \frac{\pi}{2}a_o + \left(\frac{\pi}{2} - \alpha\right)a_1 + \frac{\pi}{2}\sum_{m=1}^{\infty} \frac{a_{2m}(\cos\alpha)^{2m}}{\sum\limits_{k=o}^{m}\binom{m}{k}(-\sin^2\alpha)^k}\left\{1 + \frac{1}{(m-1)!\,2^{m-1}\sin\alpha}\right. \mathrm{x}$$

$$\left. \mathrm{x} \sum_{h=o}^{m-1} (2h-1)!!\,(2m-2h-3)!! \sum_{\ell=1}^{m-h}\binom{m}{h+\ell}(-\sin^2\alpha)^{\ell}\right\} +$$

$$+\sum_{m=1}^{\infty} a_{2m+1}(\cos\alpha)^{2m}\left\{(\frac{\pi}{2}-\alpha)\sum_{h=o}^{m}\binom{m}{h}(\tan\alpha)^{2m-2h}-\right.$$

$$\left.-\sum_{h=o}^{m-1}\sum_{\ell=1}^{m-h}\binom{m}{h}\frac{(-1)^{m-h-\ell}}{2(m-h-\ell)+1}(\tan\alpha)^{2\ell-1}\right\}, \qquad (12)$$

$$\ln r^4 = a_o \ln\left|\frac{1-\xi}{1+\xi}\right| + a_1 \ln(1-\xi^2) + \sum_{m=1}^{\infty} a_{2m}\left\{\ln\left|\frac{1-\xi}{1+\xi}\right| + 2\sum_{h=1}^{m}\frac{\xi^{2h-1}}{2h-1}\right\} +$$

$$+\sum_{m=1}^{\infty} a_{2m+1}\left\{\ln(1-\xi^2)+2\sum_{h=1}^{m}\frac{\xi^{2h}}{2h}\right\}, \qquad (13)$$

where

$$(2h-1)!! = 1x3x5x\ldots x(2h-1) \quad ,$$

is the semi-factorial of $(2h-1)$ (in particular $(-1)!!=1$).

Huynen (1958) derived an expression equivalent to (13), for the case in which $f(\xi)$ is a polynomial of degree n in ξ ; he also gave the first few terms of expansions (11) and (12).

The coefficients a_{2m} and a_{2m+1} cannot be arbitrarily chosen; in fact, the angle θ must obviously be positive for the entire range $0 \leqslant \alpha < \frac{\pi}{2}$. Suppose that $f(\xi)$ is a polynomial of degree n in ξ , then the fundamental limitation on the choice of the coefficients is:

$$\theta(a_o, a_1, \ldots, a_n; \alpha) > 0, \text{ for all } 0 \leqslant \alpha < \frac{\pi}{2} \quad , \qquad (14)$$

i.e. the representative point $(a_o, a_1, \ldots, a_n)$ must belong to a certain region of the (n+1)-dimensional space $(a_o, a_1, \ldots, a_n)$.

From formulas (2) and (3) it is seen that the relationship between N and r remains unchanged when both ξ and $f(\xi)$ change sign, i.e. when ξ and all the coefficients a_{2m} change sign (see also formula (13)). The ambiguity that thus arises in the choice of the representative point $(a_o, a_1, \ldots)$ for a given lens $N=N(r)$ is eliminated by requiring that inequality (14) be satisfied.

In order to elucidate this concept, let us examine in detail the case of formula (7), for which relation (12) reduces to

$$\theta = \frac{\pi}{2} a_o + (\frac{\pi}{2} - \alpha) a_1 \; ; \tag{15}$$

then relation (14) gives:

$$a_o + a_1 > 0, \quad a_o \geq 0 \; , \tag{16}$$

and therefore the representative point (a_o, a_1) must belong to the shaded region of Fig. 2.

If we also require that N(0) be finite and non-zero and that $(dN/dr)_{r=0} = 0$, then the representative point must belong either to the straight line DGF for which

$$a_o + a_1 = 2, \quad N(0) = 4^{a_o/4} \; , \tag{17}$$

or to the straight line DS for which

$$a_1 - a_o = 2 \; , \quad N(0) = 4^{-a_o/4} \; . \tag{18}$$

In Fig. 2, the point D(0, 2) corresponds to the free space lens ($N=1$, $\delta=0$), the point G(1, 1) to the Luneberg lens ($N = \sqrt{2-r^2}$; Luneberg, 1944) and the point F(2, 0) to Maxwell's fish eye ($N=2/(1+r^2)$), while the point H(2, 2) represents

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the isotropic lens ($N = \sqrt{(2-r)/r}$, $\delta = \pi$; see e.g. Eaton, 1953) .

Let us now impose the further restriction that the optical rays make less than one complete turn around the origin, i.e. that $\theta < 2\pi$; then the representative points (a_o, a_1) cannot be chosen outside the rhomboid $0AB_oC$ of Fig. 3. The permissable points of Fig. 2, which are external to the rhomboid $0AB_oC$ of Fig. 3 correspond to the so-called higher-order lenses, which were first considered by Stettler (1955).

If we require that all incident rays make at least ℓ, but less than $(\ell+1)$, complete turns around the origin before leaving the lens (ℓ is any positive integer), then $2\pi\ell \leqslant \theta < 2\pi(\ell+1)$, and therefore the representative point (a_o, a_1) must belong to the rhomboid $B_{\ell-1}A_\ell B_\ell C_\ell$, which is obtained by shifting $0AB_oC$, an amount 4ℓ along the positive a_o axis (see Fig. 3).

That portion of the permissible region of Fig. 2 which is not covered by the rhomboids of Fig. 3 represents lenses for which the number of complete turns of the optical ray around the origin may be changed by varying the angle α of incidence.

The investigation that we developed for the simple case of formula (7) could be repeated for the more complicated cases in which $f(\xi)$ is a polynomial of degree $n = 2, 3$, etc. In the case $n = 2$, the angle θ is given by the relation:

$$\theta = \frac{\pi}{2} a_o + (\frac{\pi}{2} - \alpha)a_1 + \frac{\pi}{2}(1-\sin\alpha)a_2 \quad , \tag{19}$$

which follows from (12).

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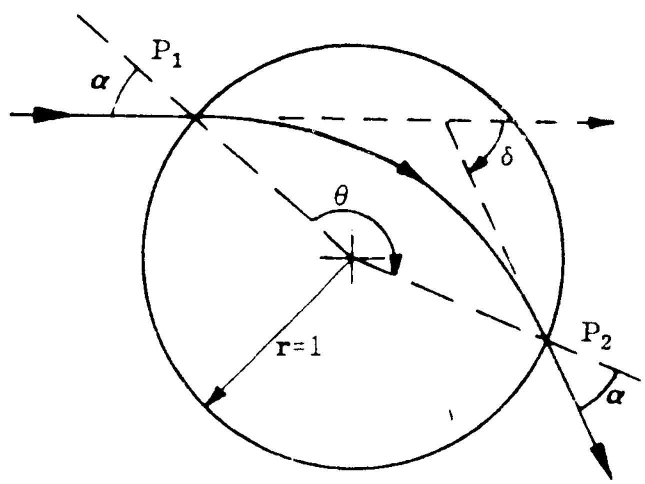
Captions for Figures

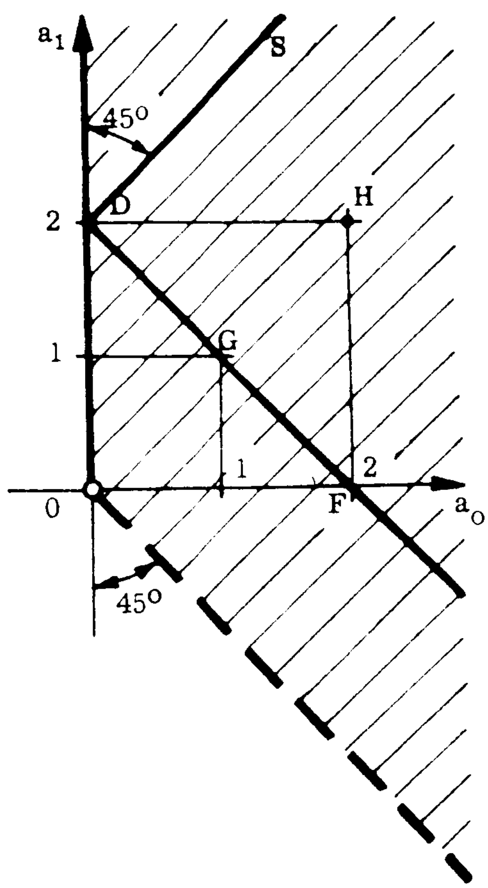
1 Geometry for the Optical Ray Path

2 The Permissible Domain in the (a_o, a_1) Plane

3 Fundamental Lenses (shaded area) and Higher-Order Lenses



P1
α
θ
δ
r=1
P2
α



a_1
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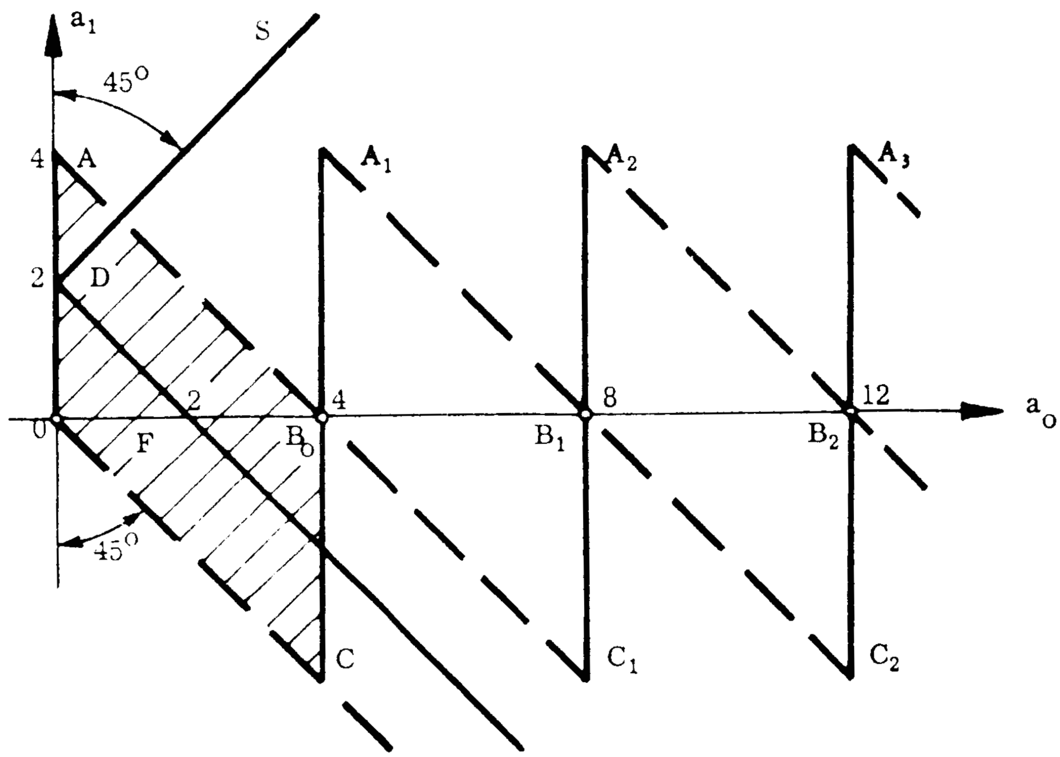
45^o
D
H
2
1
G
1
2
0
F
a_o
45^o



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45^o
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A_1
A_2
A_3
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D
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2
4
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12
a_o
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B_o
B_1
B_2
45^o
C
C_1
C_2